AP Calculus BC Free-Response Questions

AP Calculus Free-Response Questions

301. Let R be the region of the first quadrant bounded by the x-axis and the curve $y = kx - x^2$, where $k > 0$.
   (a) In terms of k, find the volume produced when R is revolved around the x-axis.
   (b) In terms of k, find the volume produced when R is revolved around they-axis.
   (c) Find the value of k for which the volumes found in (a) and (b) are equal.

302. (a)Write the first three nonzero terms in the Taylor series expansion of cos(x) about $x = \frac{\pi}{2}$.
   (b) What is the interval of convergence of the Taylor series mentioned in (a)? Show your method.
   (c) Estimate the accuracy of the approximation found in ©. Show your method.

303. Determine whether or not $\int_0^\infty xe^{-x}dx$ converges. If it converges, give the value. Show your reasoning.

304. Find a function f that has a continuous derivative on (0,∞) and that has <u>both</u> of the following properties.
   (i) The graph of f goes through the point (1,1).
   (ii) The length l of the curve from (1,1) to any point (x, f(x)) is given by the formula L = ln x + f(x) - l.
   Definition: A function f, defined for all positive real numbers, is "tractible" if and only if for every e > 0, there exists an integer M > 0 such that x > M implies that | f(x) - x | < e.
   (a) If h(x) = x - 3 for all positive numbers x, prove that h is NOT tractible.
   (b) If g(x) = $x + \frac{1}{x^2}$ for all numbers x, prove that g is tractible.
   (c) Discuss the graphical significance of tractibility.

306. (a) Does the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{3k+5}}$ converge? Justify your answer.

(b) Determine all values of x for which the series $\sum_{k=1}^{\infty} \frac{2^k x^k}{k}$ converges. Justify your answer.

307. (a) Determine the general solution of the differential equation $y'' - 4y' + 4y = e^{2x}$.

(b) Assume that xe^x is a solution of the differential equation $y'' + ay' + by = 0$. Determine the constants a and b.

308. Consider the function f defined f(x) = $\begin{cases} \frac{x}{\ln x} & \text{if } x > 0, \\ 1, & \text{if } x = 0 \\ \frac{-x}{\ln(-x)} & \text{if } x < 0. \end{cases}$

(a) For what values of x is f continuous?
(b) Is the graph of f symmetric with respect to (i) the y-axis? (ii) the origin?
(c) Find the coordinates of all relative maximum points.
(d) Find the coordinates of all relative minimum points.

309. Let C be the curve defined from t = 0 to t = 6 by the parametric equations $x = \frac{t+2}{2}, y = t(6-t)$.

Let R be the region bounded by C and the x-axis. Set up but <u>do not evaluate</u> an integral expression in terms of a single variable for
(a) the length of C
(b) the volume of the solid generated by revolving R about the x-axis
(c) the surface area of the solid generated by revolving R about the x-axis

310. A kite flies according to the parametric equations $x = \frac{t}{8}, y = \frac{-3}{64}t(t-128)$ where t is measured in seconds and $0 < t \leq 90$.
(a) How high is the kite above the ground at t = 32 seconds?
(b) At what rate is the string being reeled out at t = 32 seconds?
(c) At what rate is the kite rising at t = 32 seconds?
(d) At what time does the kite start to lose altitude?

311. The distance x of a particle from a fixed point x = 0 is given by the differential equation
$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 12x = 0$ for all time $t \geq 0$.
(a) Find the general solution of this differential equation.
(b) Find the solution of the differential equation satisfying x = 1 and $\frac{dx}{dt} = 2$. when t = 0.
(c) Does the particle pass through the point x = 0 after it begins moving? If so, determine the value of t at each such time; if not, explain.

312. In each of the following cases, decide whether the infinite series converges. Justify your answers.

$$(a)\sum_{k=1}^{\infty}(1+\frac{1}{k})^k \qquad (b)\sum_{k=1}^{\infty}\frac{\sin(k)}{k^2+\sqrt{k}} \qquad (c)\sum_{k=3}^{\infty}\frac{1}{k\ln^2 k}$$

313. Given the definite integral I = $\int_0^{\pi} kf(\sin x)\,dx$, where f is a continuous function on the closed interval [0, 1].

(a) Use the substitution x = π - y to show that I = $\frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

(b) Evaluate $\int_0^{\pi}\frac{x+\sin x}{1+\cos^2 x}dx$

314. Given the parametric equations $x = 2(\theta - \sin\theta)$ and $y = 2(1-\cos\theta)$.

(a) Find $\frac{dy}{dx}$ in terms of θ.

(b) Find an equation of the line tangent to the graph at $\theta = \pi$.

© Find an equation of the line tangent to the graph at $\theta = 2\pi$.

Set up but do not evaluate an integral representing the length of the curve over the interval $0 \le \theta \le 2\pi$. Express the integral as a function of θ.

315. Given the differential equation $y'' + 4y' + 5y = 0$.

(a) Find the general solution of y(x).

(b) Find the solution that satisfies the conditions y(0) = 0 and y'(0) = 1.

(c) Evaluate the limit of the solution determined in (b) as x increases without bound.

316. (a) Consider a function f such that $f'(x)$ exists for all real x. Suppose that

(i) $f'(r)$ = 0 and $f'(s)$ = 0,

(ii) $f'(x) \neq$ 0 for all x in the interval r < x < s; that is, r and s are consecutive roots of f '(x) = 0.

Prove that there is at most one root of f(x) = 0 in r < x < s.

(b) Given a function f such that $f'(x) = (4-x^2)(4+x^2)e^{-x^2}$. What is the maximum number of real roots of f(x) = 0 in –2 < x < 2.

317. A particle moves on the circle $x^2 + y^2 + 1$ so that at time ≥ 0 the position is given by the vector $\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}$.

(a) Find the velocity vector.

(b) Is the particle ever at rest? Justify your answer.

(c) Give the coordinates of the point that the particle approaches as t increases without bound.

318. (a) Determine whether the series $\frac{1}{3} - \frac{2^3}{3^2} + \frac{3^2}{3^2} - \frac{4^3}{3^4} + \ldots + \frac{(-1)^{n-1}n^3}{3^n} + \ldots$ is convergent.

Justify your answer.

(b) Find the interval of convergence for the series $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n(3^n)}$. Justify your answer.

319. Given the differential equation $y'' - (p+1)y' + py = 0$, where p is a real number <u>not</u> equal to 1.

(a) Find the general solution of the differential equation.

(b) Find the particular solution satisfying the conditions y(0) = 0 and $y'(0) = 1$.

(c) Find the limit of the solution in (b) as $\pi \to 1$. Justify your answer.

320. (a) For what value of m is the line y = mx tangent to the graph of y = ln x?

(b) Prove that the graph of y = ln x lies entirely below the graph of the line found in (a).

(c) Use the results of (b) to show that $e^x \geq x^e$ for x > 0.

321. A particle moves on the x-axis in such a way that its position at time t is given by $x = (2t - 1)(t - 1)^2$.

a. At what time t is the particle at rest? Justify your answer.

b. During what interval of time is the particle moving to the left? Justify your answer.

c. At what time during the interval found in (b) is the particle moving most rapidly (that is, the <u>Speed</u> is a maximum)? Justify your answer.

322. (a) Determine $\int x^2 e^{5x}\, dx$.

(b) Using integration by parts, derive a general formula for $\int x^n e^{kx}\, dx$, $k \neq 0$, in which the resulting integral involves x^{n-1}.

323. (a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3y = 0$.

(b) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3y = 1 - e^{-x}$.

(c) Find the solution of the differential equation in (b) that satisfies the conditions that y = 1 and $\frac{dy}{dx} = 0$ when x = 0.

324. (a) Write the first three nonzero terms and the general term of the Taylor series expansion about x = 0 of the function $f(x) = 5\sin(\frac{x}{2})$.

(b) What is the interval of convergence for the series found in (a)? Show your method.

(c) What is the minimum number of terms of the series in (a) that are <u>necessary</u> to approximate f(x) on the interval (-2,2) with an error not exceeding 0.1? Show your method.

325. Let f be the function defined by $f(x) = \frac{x}{1+x^2}$ for all real numbers x.

(a)

(i) Find the zeros of f.

(ii) Investigate the symmetry of the graph of f with respect to the x-axis, the y-axis, and the origin.

(iii) Find the x- and y-coordinates of the relative maximum and minimum points of f. Justify your answer.

(iv) Describe the behavior of the graph for large | x |.

(v) Using the information found above, sketch the graph of f on the axis provided.

(b) Find the area of the region bounded by the graph of f, the x-axis, and the lines $x = \frac{1}{k}$ and x = k for k > 1.

326. (a) If $\sqrt{2}e^{2x}$ is a particular solution of the differential equation $y'' - 4y' + py = 0,$ find the value of p and the general solution of the equation.

(b) Find the particular solution of the differential equation $y'' - 4y' + 5y = 0$ that satisfies the condition that y = 0 and $y' = 1$ when $x = \frac{\pi}{2}$.

327. (a) Does the series $\sum_{j=1}^{\infty} \frac{2}{j^2} \sin\frac{\pi}{j}$ converge? Justify your answer.

(b) Express $\sum_{n\to\infty}^{n} \frac{2}{n} \sin\frac{k\pi}{n}$ as a definite integral and evaluate the integral.

328. Let f(x) = $\int_0^x \frac{1}{1+t^4}dt$ for all real numbers x.

(a) Find f(0).
(b) Find $f'(1)$.
(c) Justify that f(3) - f(1) < 1.
(d) Justify that f(x) + f(-x) = 0 for all real numbers x.

329. Given the function f defined by f(x) = x | x + 2 | for all x such that -3 $\le$ x $\le$ 1.

(a) Find the values of x in the given interval for which f is increasing. Justify your answer.
(b) For what values of x is the graph of f concave downward?
(c) On the axes provided, sketch the graph of f.
(d) Is $f'(x)$ continuous for all x in the given interval? Justify your answer.

330. (a) In the interval $0 \le x \le \frac{\pi}{2}$ find the general solution of the differential equation cot(x)$\frac{dy}{dx}$+y = csc(x). $\cot x\frac{dy}{dx} + y = \csc x$

(b) Find the solution of the differential equation in part (a) that satisfies the condition that y = 0 when $x = \frac{\pi}{3}$.

331. The power series $\sum_{n=0}^{\infty} \frac{\ln(n+1)x^n}{n+1}$ has the interval of convergence -1 $\le$ x < 1. Let f(x) be its sum.

(a) Find f (0) and f '(0).
(b) Justify that the interval of convergence is -1 $\le$ x < 1.

332. A particle moves in a plane so that at any time t, 0 $\le$ t $\le$ 1, its position is given by x = (1/4) e^{8t} - 2t and y = e^{4t}. Let C denote the path traced by the particle.

a. Find the components of the velocity vector at any time t.
b. Find the arc length of C.
c. Set up an integral, involving only the variable t, that represents the area of the surface generated by rotating C about the y-axis. Do not evaluate the integral.

333. Given the differential equation $py'' + y' - 2y = qx$.

(a) Find the general solution of the differential equation when $p = 0$ and $q = 0$.

(b) Find the general solution of the differential equation when $p = 1$ and $q = 0$.

(c) Find the general solution of the differential equation when $p = 1$ and $q = 2$.

334. Let f be the function defined by $f(x) = \frac{1}{1-2x}$.

(a) Write the first four terms and the general term of the Taylor series expansion of f(x) about x = 0.

(b) What is the interval of convergence for the series found in part (a)? Show your method.

(c) Find the value of f at x = -1/4. How many terms of the series are adequate for approximating f(-1/4) with an error not exceeding one per cent? Justify your answer.

335. A particle moves in the xy-plane so that at any time $t \geq 0$ its position (x,y) is given by $x = e^{t} + e^{-t}$ and $y = e^{t} - e^{-t}$.

(a) Find the velocity vector for any time $t \geq 0$.

(b) Find $\lim_{t \to \infty} (dy/dt)/(dx/dt)$.

(c) The particle moves on a hyperbola. Find an equation for this hyperbola in terms of x and y.

(d) On the axes provided, sketch the path of the particle showing the velocity vector for t = 0.

336. Let f be a function with domain the set of all real numbers and having the following properties:

(i) $f(x + y) = f(x)f(y)$ for all real numbers x and y.

(ii) $\lim_{h \to \infty} \frac{f(h) - 1}{h} = k$, where k is a nonzero real number.

(a) Use these properties and a definition of the derivative to show that f '(x) exists for all real numbers x.

(b) Let $f^{(n)}$ denotes the nth derivative of f. Write an expression for $f^{(n)}(x)$ in terms of f(x).

(c) Given that f(1) = 2, use the Mean Value Theorem to show that there exists a number c such that 0 < c > 3 and f '© = 7/3.

337. (a) Determine whether the series $A = \sum_{n=1}^{\infty} \frac{4n}{n^2+1}$ converges or diverges. Justify your answer.

(b) If S is the series formed by multiplying the nth term in A by the nth terms in $\sum_{n=1}^{\infty} \frac{1}{2n}$, write an expression using summation notation for S.

(c) Determine whether the series S found in part (b) converges or diverges. Justify your answer.

338. (a) Find the general solution of the differential equation $xy' + y = 0$.

(b) Find the general solution of the differential equation $xy' + y = 2x^2y$.

(c) Find the particular solution for the differential equation in part (b) that satisfies the condition that $y = e^2$ when x = 1.

339. Let R be the region enclosed by the graph of $y = e^{-x}$, x = k (k > 0) and the coordinate axes.

(a) Write an improper integral that represents the limit of the area of the region R as k increases without bound and find the value of the integral if it exists.

(b) Find the volume, in terms of k, of the solid generated if R is rotated about the <u>y-axis.</u>

(c) Find the volume, in terms of k, of the solid whose base is R and whose cross sections perpendicular to the x-axis are square

.340.

Let S be the series $\sum_{n=0}^{\infty} \frac{t}{(1+t)^n}$ where $t \neq 0$.

340. (a) Find the value to which S converges when t = 1.

(b) Determine the value of t for which S converges. Justify your answer.

(c) Find all values of t that make the sum of the series S greater than 10.

341. (a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x$.

(b) Find the particular solution for the differential equation in part (a) that satisfies the conditions that y = 1 and dy/dx = -1 when x = 0.

(c) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^x$.

342. (a) A solid is constructed so that it has a circular base of radius r centimeters and every plane section perpendicular to a certain diameter of the base is a square, with a side of the square being a chord of the circle. Find the volume of the solid.

(b) If the solid described in part (a) expands so that the radius of the base increases at a constant rate of $\frac{1}{2}$ centimeters per minute, how fast is the volume changing when the radius is 4 centimeters?

343. Let f be a differentiable function defined for all x > 0 such that

(i) f(1) = 0,

(ii) f '(1) = 1, and

(iii) $\frac{d}{dx}$[f(2x)] = f '(x) , for all x > 0.

(a) Find f '(2).

(b) Suppose f ' is differentiable. Prove that there is a number c, 2 < c < 4, such that f ''(c) = -1/8.

(c) Prove that f(2x) = f(2) + f(x) for all x > 0.

344. A particle moves along the x-axis so that its position function x(t) satisfies the differential equation $\frac{d^2y}{dt^2} - \frac{dx}{dt} - 6x = 0$, and has the property that at time t = 0, x = 2 and $\frac{dx}{dt}$ = -9.

(a) Write an expression for x(t) in terms of t.

(b) At what times t, if any, does the particle pass through the origin?

(c) At what times t, if any, is the particle at rest?

345. (a) Write the Taylor series expansion about x = 0 for f(x) = ln (1 + x).
Include an expression for the general form.

(b) For what values of x does the series in part (a) converge?

(c) Estimate the error in evaluating ln (3/2) by using only the first five nonzero terms of the series in part (a). Justify your answer.

(d) Use the result found in part (a) to determine the logarithmic function whose Taylor series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{2n}$.

346. Point P(x,y) moves in the xy-plane in such a way that $\frac{dx}{dt} = \frac{1}{t+1}$ and $\frac{dy}{dt} = 2t$ for t $\geq$ 0.

(a) Find the coordinates of P in terms of t if, when t = 1, x = ln 2 and y = 0.

(b) Write an equation expressing y in terms of x.

(c) Find the average rate of change of y with respect to x as t varies from 0 to 4.

(d) Find the instantaneous rate of change of y with respect to x when t = 1.

347. Let f be the function given by f (x) = {x^2 sin (1/x), for x $\neq 0$, or 0 for x = 0}

(a) Using the definition of the derivative, prove that f is differentiable at x = 0.

(b) Find f '(x) for x = 0.

(c) Show that f ' is not continuous at x = 0.

348. Consider the curve y = 2 $x^{(1/2)}$ from x = 3 to x = 8.

a. Set up, but do not integrate, an integral expression in terms of a single variable for the length of the curve.

b. Let S be the surface generated by revolving the curve about the x-axis. Find the area of S by setting up and integrating a definite integral.

349. Consider the differential equation $\frac{dy}{dx} + 2xy = xe^{(-x^2+x)}$.

(a) Find the general solution of the differential equation.

(b) Find the particular solution of the differential equation that satisfies the condition y = 3 when x = 0.

350. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where a_0 = 1 and $a_n = \frac{7}{n} a_{n-1}$ for n $\geq$ 1.

(a) Find the first four terms and the general term of the series.

(b) If f (x) = $\sum_{n=0}^{\infty} a_n x^n$, find the value of f '(1).

351. The path of a particle is given for time t > 0 by the parametric equations $x = t + \frac{2}{t}$ and y = $3t^2$.

(a) Find the coordinates of each point on the path where the velocity of the particle in the x direction is zero.

(b) Find $\frac{dy}{dx}$ when t = 1.

(c) Find $\frac{d^2y}{dx^2}$ when y = 12.

352. Let f be the function defined by $\sum_{n=1}^{\infty} \frac{x^n n^n}{3^n n!}$ for all values of x for which the series converges.

(a) Find the radius of convergence of this series.
(b) Use the first three terms of this series to find an approximation of f(-1).
(c) Estimate the amount of error involved in the approximation in part (b). Justify your answer.

353. Consider the curves r = 3 cos θ and r = 1 + cos θ.
(a) Sketch the curves on the axes provided.
(b) Find the area of the region inside the curve r = 3 cos θ and outside the curve r = 1 + cos θ by setting up and evaluating a definite integral. Your work must include an antiderivative.

354. Given the differential equation $\frac{dy}{dx} = \frac{-xy}{\ln y}$, y > 0.

a. Find the general solution of the differential equation.
b. Find the solution that satisfies the conditon y = e^2 when x = 0. Express your answer in the form y = f(x).
c. Explain why x = 2 is not in the domain of the solution found in part (b).

355. Let f be the function defined by f(x) = - ln x for 0 < x ≤ 1 and let R be the region between the graph of f and the x-axis.
a. Determine whether region R has finite area. Justify your answer.
b. Determine whether the solid generated by revolving region R about the y-axis has finite volume. Justify your answer.

356. Let f be a function that is defined and twice differentiable for all real numbers x and that has the following properties:
(i) f (0) = 2
(ii) f ′(x) > 0 for all x
(iii) The graph of f is concave up for all x > 0 and concave down for all x < 0 Let g be the function defined by g(x) = f(x^2).
(a) Find g(0).
(b) Find the x-coordinates of all minimum points of g. Justify your answer.
(c) Where is the graph of g concave up? Justify your answer.
(d) Using the information found in parts (a), (b), and (c), sketch a possible graph of g on the axes provided.

357. Given the differential equation $\frac{dy}{dx} = 2y - 5\sin x$.

(a) Find the general solution.
(b) Find the particular solution whose tangent line at x = 0 has slope 7.

358. (a) Find the first four nonzero terms in the Taylor series expansion about x = 0 for $f(x)=\sqrt{1+x}$.

(b) Use the results found in part (a) to find the first four nonzero terms in the Taylor series expansion about x = 0 for $g(x)=\sqrt{1+x^3}$.

(c) Find the first four nonzero terms in the Taylor series expansion about x = 0 for the function h such that $h'(x)=\sqrt{1+x^3}$ and h(0)=4.

359. For all real numbers x and y, let f be a function such that f(x + y) = f(x) + f(y) + 2xy and such that $\lim \frac{f(h)}{h}=7$ as $h \to 0$.

(a) Find f(0). Justify your answer.

(b) Use the definition of the derivative to find f '(x).

(c) Find f(x).

360. At any time t $\geq$ 0, in days, the rate of growth of a bacteria population is given by y '= ky, where k is a constant and y is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days.

(a) Write an expression for y at any time t $\geq$ 0.

(b) By what factor will the population have increased in the first 10 days?

(c) At what time t, in days, will the population have increased by a factor of 6?

361. Consider the curve given by the equation $y^3 + 3x^2y + 13 = 0$.

(a) Find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point (2,-1).

(c) Find the minimum y-coordinate of any point on the curve. Justify your answer.

362. Let R be the region enclosed by the graph of y = ln x, the line x = 3, and the x-axis.

a. Find the area of the region R.

b. Find the volume of the solid generated by revolving region R about the x-axis.

c. Set up, but <u>do not integrate</u>, an integral expression in terms of a single variable for the volume of the solid generated by revolving region R about the line x = 3.

363. (a) Find the first five terms in the Taylor series about x = 0 for $\frac{1}{1-2x}$.

(b)Use partial fractions and the result from part (a) to find the first five terms in the Taylor series about x = 0 for $g(x)=\frac{1}{(1-2x)(1-x)}$.

364. The position of a particle moving in the xy-plane at any time t, $0 \leq t \leq 2\pi$, is given by the parametric equations x = sin t and y = cos (2t).

a. Find the velocity vector for the particle at any time t, $0 \leq t \leq 2\pi$.

b. For what values of t is the particle at rest?

c. Write an equation for the path of the particle in terms of x and y that does not involve trigonometric functions.

(d) Sketch the path of the particle in the xy-plane below.

365. Let f be a continuous function with domain x > 0 and let F be the function given by F(x) = $\int_t^x f(t)dt$ for x > 0.

Suppose that F(ab) = F(a) + F(b) for all a > 0 and b > 0 and that F '(1) = 3.

(a) Find f(1).

(b) Prove that af '(ax) = F '(x) for every positive constant a.

(c) Use the results from parts (a) and (b) to find f(x). Justify your answer.

366. Let f be the function defined by f(x) = $(x^2 - 3)e^x$ for all real numbers x.

(a) For what values of x is f increasing?

(b) Find the x-coordinate of each point of inflection of f.

(c) Find the x- and y-coordinates of the point. If any, where f(x) attains its absolute minimum.

367. Let R be the shaded region between the graphs of $y = \frac{3}{x}$ and $y = \frac{3x}{x^2+1}$ for x = 1 to x = $\sqrt{3}$.

(a) Find the area of R.

(b) Find the volume of the solid generated by revolving R about the y-axis.

368.

The figure above represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from point C.

(a) Find the rate of change in x at the instant when y = 50.

(b) Find the rate of change in the area of right triangle BCA at the instant when y = 50.

(c) Find the rate of change in θ at the instant when y = 50.

369. Determine all values of x for which the series $\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$ converges. Justify your answer.

370. The base of a solid S is the shaded region in the xy-plane enclosed by the x-axis, the y-axis, and the graph of y = 1 - sin x, as shown in the figure above. For each x, the cross section of S perpendicular to the x-axis at the point (x, 0) is an isosceles right triangle whose hypotenuse lies in the xy-plane.

a. Find the area of the triangle as a function of x.

b. Find the volume of S.

371. Let f be a differentiable function defined for all x $\geq$ 0 such that f (0) = 5 and f(3) = -1. Suppose that for any number b > 0 the average value of f(x) on the interval 0 $\leq$ x $\leq$ b is $\frac{f(0)+f(b)}{2}$

(a) Find $\int_0^3 f(x)\,dx$

(b) Prove that $f'(x) = \frac{f(x)-5}{x}$ for all x > 0.

(c) Using part (b), find f(x).

372. Let f be a function such that $f''(x) = 6x + 8$.

(a) Find f(x) if the graph of f is tangent to the line $3x - y = 2$ at the point (0,-2).

(b) Find the average value of f(x) on the closed interval [-1, 1].

373. Let R be the region enclosed by the graph of $\frac{x^2}{x^2+1}$, the line x = 1, and the x-axis.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the y-axis.

374. Consider the function f defined by $f(x) = e^x \cos x$ with domain $[0, 2\pi]$.

(a) Find the absolute maximum and minimum values of f(x).

(b) Find the intervals on which f is increasing.

(c) Find the x-coordinate of each point of inflection of the graph of f.

375. Consider the curve given by the parametric equations $x = 2t^3 - 3t^2$ and $y = t^3 - 12t$.

(a) In terms of t, find $\frac{dy}{dx}$.

(b) Write an equation for the line tangent to the curve at the point where t = -1.

(c) Find the x- and y-coordinates for each critical point on the curve and identify each point as having a vertical or horizontal tangent.

376. At any time $t \geq 0$, the velocity of a particle traveling along the x-axis is given by the differential equation $\frac{dx}{dt} - 10x = 60e^{4t}$.

(a) Find the general solution x(t) for the position of the particle.

(b) If the position of the particle at time t = 0 is x = -8, find the particular solution x(t) for the position of the particle.

(c) Use the particular solution from part (b) to find the time at which the particle is at rest.

377. Let f be a function that is everywhere differentiable and that has the following properties:

(i) f (x + h) = [(f(x) + f(h)) / (f(-x) + f(-h))] for all real numbers h and x.

(ii) f (x) > 0 for all real numbers x.

(iii) $f'(0) = -1$.

(a) Find the value of f(0).

(b) Show that f(-x) = 1 / f(x) for all real numbers x.

(c) Using part (b), show that f(x + h) = f(x) f(h) for all real numbers h and x.

(d) Use the definition of the derivative to find $f'(x)$ in terms of f(x).

378. A particle starts at time t = 0 and moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = (t - 1)^3(2t - 3)$.

(a) Find the velocity of the particle at any time $t \geq 0$.

(b) For what values of t is the velocity of the particle less than zero?

(c) Find the value of t when the particle is moving and the acceleration is zero.

379. Let R be the region in the xy-plane between the graphs of $y = e^x$ from x = 0 to x = 2.

(a) Find the volume of the solid generated when R is revolved about the x-axis.

(b) Find the volume of the solid generated when R is revolved about the y-axis.

380. Let $f(x) = 12 - x^2$ for $x \geq 0$ and $f(x) \geq 0$.

a. The line tangent to the graph of f at the point (k, f(k)) intercepts the x-axis at x = 4. What is the value of k?

b. An isosceles triangle whose base is the interval from (0,0) to (c,0) has its vertex on the graph of f. For what value of c does the triangle have maximum area? Justify your answer.

381. Let R be the region inside the graph of the polar curve r = 2 and outside of the polar curve $r = 2\ (1 - \sin\theta)$.

(a) Sketch the two polar curves in the xy-plane provided and shade the region R.

(b) Find the area of R.

382. Let f be the function defined by $f(x) = \dfrac{1}{x-1}$.

a. Write the first four terms and the general term of the Taylor series expansion of f(x) about x = 2.

b. Use the result from part (a) to find the first four terms and the general term of the series expansion about x = 2 for ln | x - 1 |.

c. Use the series in part (b) to computer a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

383. Let f and g be continuous functions with the following properties:

(i) g(x) = A - f(x) where A is a constant

(ii) $\int_1^2 f(x)dx = \int_2^3 g(x)dx$

(iii) $\int_2^3 f(x)\ dx = -3A$

(a) Find $\int_1^3 f(x)dx$ in terms of A.

(b) Find the average value of g(x) in terms of A, over the interval [1,3].

(c) Find the value of k if $\int_0^1 f(x+1)\,dx = kA$.

384. A particle moves on the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 12t^2 - 38t + 15$. At t = 1, the particle is at the origin.

(a) Find the position x(t) of the particle at any time $t \geq 0$.

(b) Find all values of t for which the particle is at rest.

(c) Find the maximum velocity of the particle for $0 \leq t \leq 2$.

(d) Find the total distance traveled by the particle from t = 0 to t = 2.

385. Let f be the function defined by $f(x) = x\,e^{(1 - x)}$ for all numbers x.

(a) Find each interval on which f is increasing.

(b) Find the range of f.

(c) Find the x-coordinate of each point of inflection of the graph of f.

(d) Using the results found in parts (a), (b), and (c), sketch the graph of f in the xy-plane provided. (Indicate all intercepts.)

386. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of y = sin x and y = cos x.

(a) Find the area of R.
(b) Find the volume of the solid generated when R is revolved about the x-axis.
(c) Find the volume of the solid whose base is R and whose cross sections cut by planes perpendicular to the x-axis are squares.

387. Let $F(x) = \int_1^{2x} \sqrt{t^2 + 1}\, dt.$

(a) Find F'(x).
(b) Find the domain of F.
(c) Find $\lim_{x \to .5} F(x)$.
(d) Find the length of the curve y = F(x) for $1 \le x \le 2$.

388. Let F be the function given by $f(t) = 4 / (1 + t^2)$ and G be the function given by $G(x) = \int_0^x f(t)\, dt$.

a. Find the first four nonzero terms and the general term for the power series expansion of f(t) about t = 0.
b. Find the first four nonzero terms and the general term for the power series expansion of G(x) about x = 0.
c. Find the interval of convergence of the power series in part (b). (Your solution must include an analysis that justifies your answer.)

389. A certain rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(1 - y)$, where y is the proportion of the population that has heard the rumor at time t.

(a) What proportion of the population has heard the rumor when it is spreading the fastest?
(b) If at time t = 0 ten percent of the people have heard the rumor, find y as a function of t.
(c) At what time t is the rumor spreading the fastest?

390. At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the parametric equations $x = e^t \sin t$ and $y = e^t \cos t$.

(a) Find the slope of the path of the particle at time $t = \pi/2$.
(b) Find the speed of the particle when t = 1.
(c) Find the distance traveled by the particle along the path from t = 0 to t = 1.

391. Let f be a function defined by $f(x) = \begin{cases} 2x - x^2 & \text{for } x \le 1, \\ x^2 + kx + p, & \text{for } x > 1. \end{cases}$

a. For what values of k and p will f be continuous and differentiable at x = 1?
b. For the values of k and p found in part (a), on what intervals is f increasing?
c. Using the values of k and p found in part (a), find all points of inflection of the graph of f. Support your conclusion.

392. The length of a solid cylindrical cord of elastic material is 32 inches. A circular cross section of the cord

has radius ½ inch.

a. What is the volume, in cubic inches, of the cord?

b. The cord is stretched lengthwise at a constant rate of 18 inches per minute. Assuming that the cord maintains a cylindrical shape and a constant volume, at what rate is the radius of the cord changing one minute after stretching begins. Indicate units of measure. A force of 2x pounds is required to stretch the cord x inches beyond its natural length of 32 inches.

c. How much work is done during the first minute of stretching described in part (b)? Indicate units of measure.

393. Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$, where $p \geq 0$.

(a) Show that the series converges for $p > 1$.

(b) Determine whether the series converges or diverges for $p = 1$. Show your analysis.

(c) Show that the series diverges for $0 \leq p < 1$.

394. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y = (2/3)\, t^3$.

(a) Find the magnitude of the velocity vector at $t = 5$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(c) Find dy/dx as a function of x.

396. Let f be the function given by $f(x) = e^{(x/2)}$.

a. Write the first four nonzero terms and the general term for the Taylor series expansion for f(x) about x = 0.

b. Use the result from part (a) to write the first three nonzero terms and the general terms of the series expansion about x = 0 for $g(x) = (e^{x/2} - 1)/x$.

c. For the function g in part (b), find g′(2) and use it to show that $\sum_{n=1}^{\infty} \frac{n}{4(n+1)!} = \frac{1}{4}$.

397. Let f be a function that is differentiable throughout its domain and that has the following properties:

(i) $f(x + y) = [f(x) = f(y)] / [1 - f(x)\, f(y)]$ for all real numbers x, y, and x + y in the domain of f.

(ii) $\lim_{h \to 0} f(h) = 0$.

(iii) $\lim_{h \to 0} f(h)/h = 1$.

(a) Show that $f(0) = 0$.

(b) Use the definition of the derivative to show that $f'(x) = 1 + [f(x)]^2$.
Indicate clearly where properties (i), (ii), and (iii) are used.

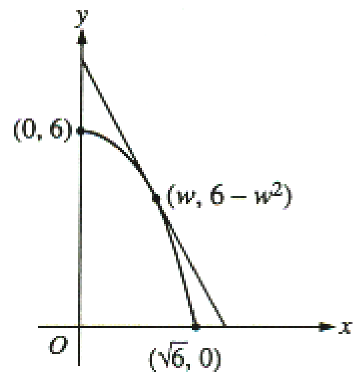
(c) Find f(x) by solving the differential equation in part (b).

398. A particle moves along the graph of $y = \cos x$ so that the x-component of acceleration is always 2. At time t = 0, the particle is at the point $(\pi, -1)$ and the velocity vector of the particle is (0,0).

(a) Find the x- and y-coordinates of the position of the particle in terms of t.

(b) Find the speed of the particle when its position is (4, cos 4).

399. Let $f(x) = 6 - x^2$. For $0 < w < \sqrt{6}$, let $A(w)$ be the area of the triangle formed by the coordinate axes and the line tangent to the graph of f at the point $(w, 6 - w^2)$. (see figure below.)



Note: Figure not drawn to scale.

(a) Find A(1).
(b) For what value of w is A(w) a minimum?

400. Let f be the function given by $f(x) = e^{-2x^2}$

a. Find the first four nonzero terms and the general term of the power series for f(x) about x = 0.
b. Find the interval of convergence of the power series for f(x) about x = 0. Show the analysis that leads to your conclusion.
c. Let g be the function given by the sum of the first four nonzero terms of the power series for f(x) about x = 0.
d. Show that $| f(x) - g(x) | < 0.02$ for $-0.6 \leq x \leq .6$.

401. Let f and g be functions that are differentiable for all real numbers x and that have the following properties:

(i) $f'(x) = f(x) - g(x)$
(ii) $g'(x) = g(x) - f(x)$
(iii) $f(0) = 5$
(iv) $g(0) = 1$
(a) Prove that $f(x) + g(x) = 6$ for all x.
(b) Find f(x) and g(x). Show your work.

402. Two particles move in the xy-plane. For time $t \geq 0$, the position of particle A is given by $x = t - 2$ and $y = (t - 2)^2$, and the position of particle B is given by $x = (3/2)t - 4$ and $y = (3/2)t - 2$.

a. Find the velocity vector for each particle at time t - 3.

b. Set up an integral expression that gives the distance traveled by particle A from t = 0 to t = 3. Do not evaluate.

c. Determine the exact time at which the particles collide; that is , when the particles are at the same point at the same time. Justify your answer.

d. In the viewing window of [-7,7] by [-5,5], sketch the paths of the particles A and B from t = 0 until they collide. Indicate the direction of each particle along its path.

403. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1) = 3$, $f'(1) = -2$, $f''(x) = 2$, and $f'''(1) = 4$.

a. Write the second-degree Taylor polynomial for f about x = 1 and use it to approximate f(0.7).

b. Write the third-degree Taylor polynomial for f about x = 1 and use it to approximate f(1.2).

c. Write the second-degree Taylor polynomial fo f ', the derivative of f, about x = 1 and use it to approximate f '(1.2).

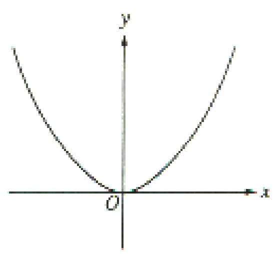


Figure 1
$y = f(x)$

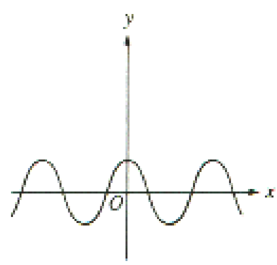


Figure 2
$y = g(x)$

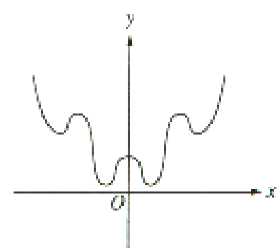


Figure 3

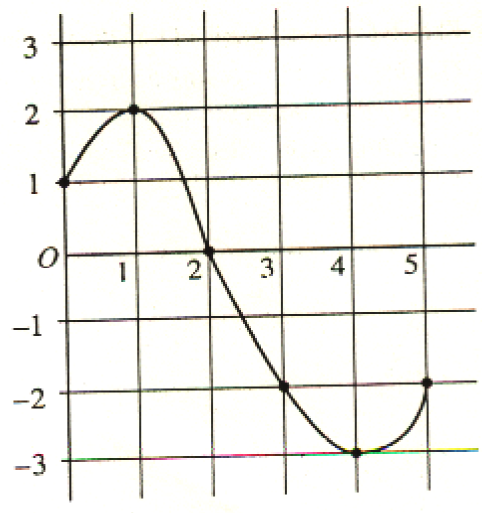
404. Let $f(x) = x^2$, $g(x) = \cos x$, and $h(x) = x^2 + \cos x$. From the graphs of f and g shown above in Figure 1 and Figure 2, one might think that the graph of h should look like the graph in Figure 3.

a. Sketch the actual graph of h in the viewing window [-6,6] by [-6, 40].

b. Use $h''(x)$ to explain why the graph of h does not look like the graph in Figure 3.

c. Prove that the graph of $y = x^2 + \cos(kx)$ has either no points of inflection or infinitely many points of inflection, depending on the value of the constant k.

405. Let f be a function whose domain is the closed interval [0,5]. The graph of f is shown below.

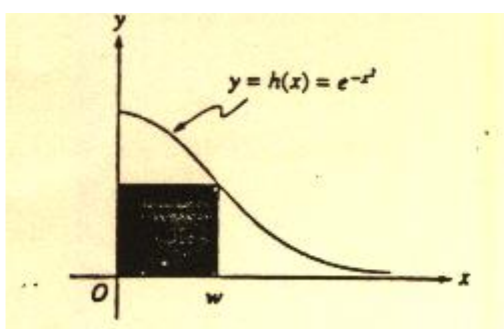


Graph of f

Let $h(x) = \int_0^{\frac{x}{2}+3} f(t)\, dt$.

(a) Find the domain of h.
(b) Find h'(2).
(c) At what x is h(x) a minimum? Show the analysis that leads to your conclusion.

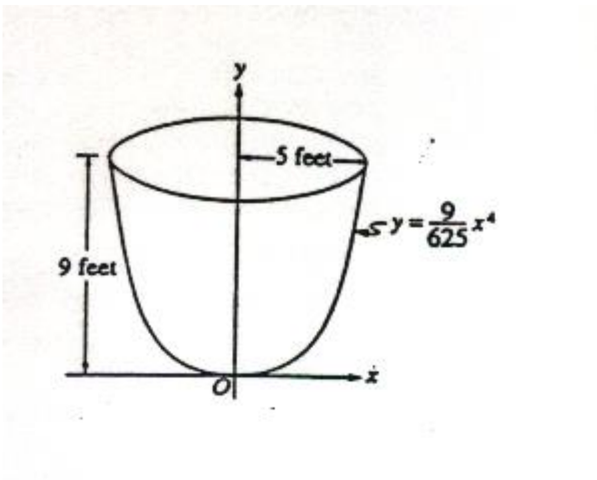
406. Consider the graph of the function h given by $h(x) = e^{-x^2}$ for $0 \le x < \infty$.



a. Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.
b. Let A(w) be the area of the shaded rectangle shown in the figure above. Show that A(w) has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.

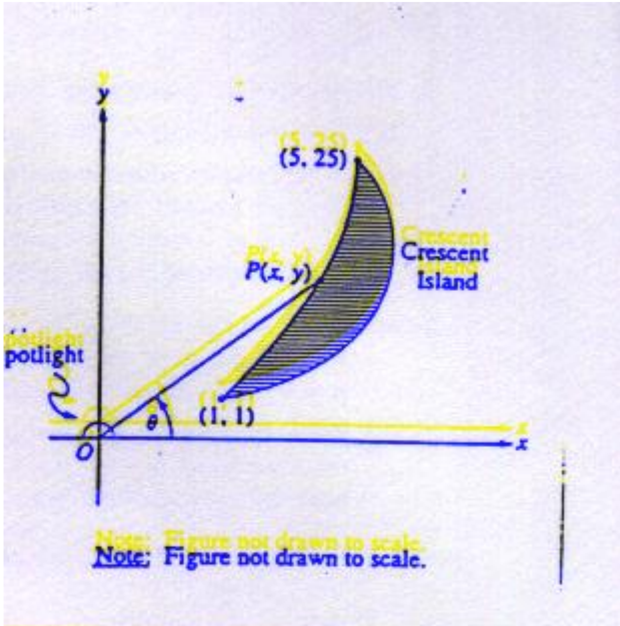
407. The Maclaurin series for f(x) is given by $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+...+\frac{x^n}{(n+1)!}+...$

(a) Find f′(0) and $f^{17}(0)$.
(b) For what values of x does the given series converge? Show your reasoning.
(c) Let g(x) = x f(x). Write the Maclaurin series for g(x), show the first three nonzero terms and the general term.
(d) Write g(x) in terms of a familiar function without using series. Then, write f(x) in terms of the same familiar function.



408. An oil storage tank has the shape shown above, obtained by revolving the curve $y=\frac{9}{625}x^4$ from x = 0 to x = 5 about the y-axis, where x and y are measured in feet. Oil weighing 50 pounds per cubic foot flowed into an initially empty tank at a constant rate of 8 cubic feet per minute. When the depth of the oil reached 6 feet, the flow stopped.

a. Let h be the depth, in feet, of oil in the tank. How fast was the depth of the oil in the tank increasing when h = 4? Indicate units of measure.
b. Find, to the nearest foot-pound, the amount of work required to empty the tank by pumping all of the oil back to the top of the tank.



409. The figure above shows a spotlight shining on point P(x,y) on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shines is in the shape of the parabola $y = x^2$ from the point (1, 1) to the point (5, 25). Let θ be the angle between the beam of light and the positive x-axis.

(a) For what values of θ between 0 and 2π does the spotlight shine on the shoreline?
(b) Find the x- and y-coordinates of point P in terms of tan θ. If the spotlight is rotating at the rate of one revolution per minute, how fast is the point P traveling along the shoreline at the instant it is at the point (3, 9)?

410. During the time period from t = 0 to t = 6 seconds, a particle moves along the path given by $x(t) = 3\cos(\pi t)$ and $y(t) = 5\sin(\pi t)$.

a. Find the position of the particle when t = 2.5.
b. On the axes provided, sketch the graph of the path of the particle from t = 0 to t = 6. Indicate the direction of the particle along its path.
c. How many times does the particle pass through the point found in part a?
d. Find the velocity vector for the particle at any time t.
e. Write and evaluate an integral expression, in terms of sine and cosine, that gives the distance the particle travels from time t = 1.25 to t = 1.75.

411. Let $P(x) = 7 - 3(x - 4) + 5(x - 4)^2 - 2(x - 4)^3 + 6(x - 4)^4$ be the fourth-degree Taylor polynomial for the function f about 4. Assume f has derivatives of all orders for all real numbers

(a) Find f(4) and $f'''(4)$.

(b) Write the second-degree Taylor polynomial for f′ about 4 and use it to approximate f′(4.3).

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\,dt$ about 4.

(d) Can f(3) be determined from the information given? Justify your answer.

412. Let R be the region enclosed by the graphs of $y = \ln(x^2 + 1)$ and $y = \cos x$.

a. Find the area of R.

b. Write an expression involving one or more integrals that give the length of the boundary of the region R. Do not evaluate.

c. The base of a solid is the region R. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle. Write an expression involving one or more integrals that gives the volume of the solid. Do not evaluate.

413. Let $x = ky^2 + 2$, where k > 0.

a. Show that for all k > 0, the point $(4, \frac{\sqrt{2}}{k})$ is on the graph of $x = ky^2 + 2$. Show that for all k > 0, the tangent line to the graph of $x = ky^2 + 2$ at the point $(4, \frac{\sqrt{2}}{k})$ passes through the origin.

b. Let R be the region in the first quadrant bounded by the x-axis, the graph of $x = ky^2 + 2$, and the line x = 4.

c. Write an integral expression for the area of the region R and show that this area decreases as k increases.

414. Let f be a function that has derivatives of all orders for all real numbers. Assume f(0) = 5, f′(0) = -3, f″(0) = 1, and $f'''(0) = 4$.

a. Write the 3rd degree Taylor polynomial for f about x = 0 and use it to approximate f(0.2).

b. Write the 4th degree Taylor polynomial for g, where $g(x) = f(x^2)$, about x = 0.

c. Write the 3rd degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about x = 0.

d. Let h be defined as in part © . Given that f(1) = 3, either find the exact value of h(1) or explain why it cannot be determined.

415. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

a. Sketch a slope field for the given differential equation at the nine points (0, 1), (0, 2), (0, 3), (-1, 1), (-1, 2), (-1, 3), (1, 1), (1, 2), (1, 3)

b. Let y = f(x) be the particular solution to the given differential equation with the initial condition f(0) = 3. Use Euler's method starting at x = 0, with a step size of 0.1, to approximate f(0.2). Show the work that leads to your answer.

c. Find the particular solution y = f(x) to the given differential equation with the initial condition f(0) = 3. Use your solution to find f(0.2).

416. A particle moves along the curve defined by the equation $y = x^3 - 3x$. The x-coordinate of the particle, x(t), satisfies the equation $\frac{dx}{dt} = \frac{1}{\sqrt{2t+1}}$, for $t \geqslant 0$ with initial condition x(0) = -4.

a. Find x(t) in terms of t.

b. Find $\frac{dy}{dt}$ in terms of t.

c. Find the location and speed of the particle at time t = 4.

417. A particle moves in the xy-plane so that its position at any time t, $0 \leqslant t \leqslant \pi$, is given by $x(t) = \frac{t^2}{2} - \ln(1+t)$ and y(t) = 3sin t.

a. Sketch the path of the particle. Indicate the direction of motion along the path.
b. At what time t, $0 \leqslant t \leqslant \pi$, does x(t) attain its minimum value? What is the position (x(t), y(t)) of the particle at this time?
c. At what time t, $0 < t < \pi$, is the particle on the y-axis? Find the speed and the acceleration vector of the particle at this time.

418. The function f has derivatives of all orders for all real numbers x. Assume f(2) = -3, f '(2) = 5, f "(2) = 3, and f '''(2) = -8.

a. Write the third-degree Taylor polynomial for f about x = 2 and use it to approximate f(1.5).
b. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 3$ for all x in the closed interval [1.5, 2]. Use the Lagrange error bound on the approximation to f(1.5) found in part a. to explain why f(1.5) ≠ -5.
c. Write the fourth-degree Taylor polynomial, P(x), for $g(x) = f(x^2 + 2)$ about x = 0. Use P to explain why g must have a relative minimum at x = 0.

419. Let f be the function whose graph goes through the point (3, 6) and whose derivative is given by $f'(x) = \frac{1+e^x}{x^2}$.

a. Write the equation of the line tangent to the graph of f at x = 3 and use it to approximate f(3.1).
b. Use Euler's method, starting at x = 3 with a step size of 0.05, to approximate f(3.1). Use f " to explain why this approximation is less than f(3.1).
c. Use $\int_3^{3.1} f'(x)\,dx$ to evaluate f(3.1).